

# On Clifford representation of Hopf algebras and Fierz identities.

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Abstract. We present a short review of the action and coaction of Hopf algebras on Clifford algebras as an introduction to physically meaningful examples. Some  $q$ -deformed Clifford algebras are studied from this context and conclusions are derived.

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# 1 INTRODUCTION.

Like a group, a Hopf algebra can act as the algebra of symmetries of another algebra (cf. the group of symmetries of a quantum system). In this paper we explore the possible quantum Clifford action (coaction) on Clifford algebras. Hopf algebras can act on other structures in a variety of ways; i.e.,  $H$  can act as an algebra on a vector space. Besides, we can think of quantization as a procedure that replaces the classical algebra of observables by a non-commutative quantum algebra of observables [1]. It is expected that even using non-commutative geometry, one might nevertheless extend our regular notions of symmetry to the quantum world. The simplest non-commutative geometries that have been studied are Clifford algebras and Hopf algebras corresponding to both quantum symmetry and curvature. Thus, it is a natural question to inquire about the way Hopf algebras act (and coact) on Clifford algebras to understand how quantum structures act (and coact) on classical ones as physically meaningful symmetries. Other attempts to link

Clifford algebra to  $q$ -deformed geometry have been reported recently. See for example [2].

Quantum algebras have already proved useful in the study of quantum spin chains and conformal field theories, and may have other important physical applications as well.

## 2 Clifford category of representations of a triangular Hopf algebra.

Let us begin this Section with some formal definitions. A *Hopf algebra*

$(H, +, \cdot, \eta, \Delta, \epsilon, S: k)$  over  $k$ , a field, is a bialgebra over  $k$  equipped with a linear antipode map  $S: H \rightarrow H$  obeying  $\cdot(S \otimes id) \circ \Delta = \cdot(id \otimes S) \circ \Delta = \eta \circ \epsilon$ .

Here  $\epsilon$  is the counit map  $\epsilon: H \rightarrow k$ ,  $\Delta$  is the coproduct map  $\Delta: H \rightarrow H \otimes H$

and  $\eta$  is the unit map  $\eta: k \rightarrow H$ . A *bialgebra*  $(B, +, \cdot, \eta, \Delta, \epsilon; k)$  over  $k$

is a vector space  $(B, +; k)$  over  $k$  which is both an algebra and a coalgebra,

in a compatible way; i.e.  $\Delta(ab) = \Delta(a)\Delta(b)$ ,  $\Delta(1) = 1 \otimes 1$ ,  $\epsilon(ab) = \epsilon(a)\epsilon(b)$  and

$\epsilon(1) = 1$ . Finally, a *coalgebra*  $(\mathcal{A}, +, \Delta, \epsilon; k)$  over  $k$  is a vector space

$(\mathcal{A}, +; k)$  over  $k$  and a linear map  $\Delta$  which is coassociative and for which

there exists a linear counit map  $\epsilon$ . In this paper we are specially interested in non-commutative and non-cocommutative Hopf algebras. This means that neither the product nor the coproduct are commutative.

A *representation* of an algebra  $H$  is a pair  $(\alpha, V)$  where  $V$  is a vector space and  $\alpha$  a linear map  $H \otimes V \rightarrow V$ , say  $\alpha(h \otimes v) = \alpha_h(v)$  for  $h \in H$  and  $v \in V$ , where  $\alpha_h(\alpha_g(v)) = \alpha_{hg}(v)$ . A left  $H$ -module is nothing other than the vector space on which the algebra  $H$  is represented.

If  $\alpha$  is an antirepresentation, namely  $\alpha_h(\alpha_g(v)) = \alpha_{gh}(v)$ , then  $\alpha$  is the right action of  $H$  on  $V$ . In other words,  $V$  is a right  $H$ -module.

The collection of left  $H$ -modules is denoted by  ${}_H M$  and the collection of right  $H$ -modules is denoted by  $M_H$ .

If  $H$  is a Hopf algebra then it can act on algebras in such a way as to respect the algebra structure. Thus,  $H$  acts on  $A$  as an algebra ( or  $A$  is an  *$H$ -module algebra*) if  $A$  is an  $H$ -module and in addition

$$\alpha(h \otimes ab) = \alpha(h_{(1)} \otimes a)\alpha(h_{(2)} \otimes b), \quad \alpha(h \otimes 1_A) = \epsilon 1_A$$

for  $a, b \in A$  and  $h \in H$ . Here  $\mathbf{1}_A$  is the unit in  $A$  and  $\Delta(h) = h_{(1)} \otimes h_{(2)}$ .

As an example of a Hopf algebra acting on a Clifford algebra, let us consider, formally,  $GL_q(2, \mathbb{C})$  acting on  $Cl_{3,1}$ . Here  $\gamma_\mu \in Cl_{3,1}$  such that  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{1}$ ;  $\mu, \nu = 0, \dots, 3$  and  $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ . On the other hand we define  $GL_q(2, \mathbb{C})$  as follows. Let  $a_{ij} \in GL_q(2, \mathbb{C})$ ;  $i, j = 1, 2$ , then  $a_{11}a_{12} = qa_{12}a_{11}$ ;  $a_{11}a_{21} = qa_{21}a_{11}$ ;  $a_{12}a_{21} = a_{21}a_{12}$ ;  $a_{12}a_{22} = qa_{22}a_{12}$ ;  $a_{21}a_{22} = qa_{22}a_{21}$  and  $a_{11}a_{22} - a_{22}a_{11} = (q - q^{-1})a_{12}a_{21}$ . Besides  $\Delta(a_{ij}) = \sum_{k=1}^2 a_{ik} \otimes a_{kj}$ ,  $\epsilon(a_{11}) = \epsilon(a_{22}) = 1$  and  $\epsilon(a_{12}) = \epsilon(a_{21}) = 0$ . Here  $q$  is any complex number. We adopt the concept of quantum group in a purely geometrical point of view [3].

We can see that  $q$  plays the role of a deformation parameter which, in the limit  $q \rightarrow \pm 1$ , includes bose and fermi statistics in Fock space. The generators  $a_{ij}$  have been identified with creation and annihilation operators in  $q$ -deformed oscillators for a wide variety of authors. See for example [4]. In this case the parameter  $q$  allows us to generalize statistics to include parfermi, infinite and other cases [5]. Some applications related to the spectra of triatomic molecules and superdeformed nuclei have also been studied.

Let

$$\alpha(a_{ij} \otimes \gamma_\mu \gamma_\nu) = \alpha(a_{ij(1)} \otimes \gamma_\mu) \alpha(a_{ij(2)} \otimes \gamma_\nu) \quad \forall \mu, \nu \quad \text{and}$$

$$\alpha(a_{ij} \otimes \gamma_\mu \gamma_\nu) = \alpha(a_{ij(1)} \otimes \gamma_\mu) \cdot \alpha(a_{ij(2)} \otimes \gamma_\nu) + \alpha(a_{ij(1)} \otimes \gamma_\mu) \wedge \alpha(a_{ij(2)} \otimes \gamma_\nu) =$$

$$\alpha(a_{ij(k)} \otimes g_{\mu\nu}) + \alpha(a_{ij(k)} \otimes \gamma_{\mu\nu})$$

for any  $\gamma_\mu, \gamma_{\mu\nu} \in \mathcal{Cl}_{3,1}$  and  $k = 1, 2$ . From this, the action of the map on any

$\Gamma \in \mathcal{Cl}_{3,1}$  follows. A concrete map remains to be found.

In this paper we present two physically meaningful concrete examples related with this matter.

1.  **$\mathcal{Cl}_{3,1}$  as a  $C_q(3,1)$ -module.** In this case we are thinking of  $C_q(3,1)$ , a  $q$ -deformed Clifford algebra, as symmetry of  $\mathcal{Cl}_{3,1}$ . Following Manin [3],  $C_q(3,1)$  can be defined by considering a space, the coordinates of which do not commute, as follows. Let  $\{\gamma_q^\mu\} \in C_q(3,1)$  then  $\gamma_q^\mu \gamma_q^\nu + q \hat{\mathbf{R}}_{\nu'\mu'}^{\mu\nu} \gamma_q^{\nu'} \gamma_q^{\mu'} =$

$q^{-1}QC^{-1\mu\nu}$ , where

$$\gamma_q^0 = \begin{pmatrix} 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \gamma_q^+ = \sqrt{qQ} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_q^- = \sqrt{Q} \begin{pmatrix} 0 & 0 & 0 & q^{-3/2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -q^{3/2} & 0 & 0 & 0 \end{pmatrix} \quad \gamma_q^3 = \begin{pmatrix} 0 & 0 & q^{-1} + q - q^2 & 0 \\ 0 & 0 & 0 & -q^{-2} \\ -1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \end{pmatrix},$$

Here  $Q = q + q^{-1}$ , being  $q$  a real number,  $\hat{\mathbf{R}}_{\nu'\mu'}^{\mu\nu}$  is the  $SO_q(3,1)$ -R matrix

and  $C$  is the following metric [6].

$$\begin{pmatrix} 0 & 0 & 0 & q^{-1} \\ 0 & -1 + q^{-2} & -q^{-1} & 0 \\ 0 & -q^{-1} & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}.$$

We define  $\alpha: \sum_\rho \sum_\mu (\gamma_q^\mu)_\nu^\rho \otimes \gamma_\mu \rightarrow \gamma_\nu$ . Before  $\alpha$  is applied  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbf{1}$

with  $\mu, \nu, \rho = 0, +, -, 3$  and  $g = [g_{\mu\nu}] = \text{diag}(-1, 1, 1, 1)$ . Here the indices

$0, +, -, 3$  correspond to the indices  $\mu$  in the  $\gamma_q^\mu$  matrices above defined. After

applying the map,  $\{\alpha_{\gamma_q^\mu}, \alpha_{\gamma_q^\nu}\} = 2\alpha_{g^{\mu\nu}}\mathbf{1}$ , where

$$\alpha_{g^{\mu\nu}} =$$

$$\begin{pmatrix} Qq^{-3} & 1 - q^2 & q - q^{-1} & 1 - Qq^{-3} + q^{-2} \\ 1 - q^2 & Qq + q^4 - 1 & q - q^2 + q^3 - q^4 & qQ \\ q - q^{-1} & q - q^2 + q^3 - q^4 & Q(Q - 2q^2) & 1 - q^2 - q^{-1} - q^{-3} \\ 1 - Qq^{-3} + q^{-2} & qQ & 1 - q^2 - q^{-1} - q^{-3} & -1 + qQ + q^{-3} + q^{-4} \end{pmatrix}.$$

From this, the  $C_q(3, 1)$ -invariance of the general Fierz identity [7] follows, which comes from the multivectorial generalization of the Cartan map [8] used to obtain a multivectorial Dirac equation [9]. In this multivectorial Dirac equation, by means of the comodule map  $\beta$  defined from the specific  $\alpha$  in this case (reversing arrows), we can show the  $C_q(3, 1)$  symmetry.

**2.  $su(2)$  as a  $CH_q(2)$ -module algebra.** Let us consider in this case the affinization of the Hopf algebra  $CH_q(2)$  acting on the  $su(2)$  algebra.  $CH_q(2)$  is the quantum deformation of the Hopf algebra  $CH(2)$  [10]. Here  $CH(2)$  is generated by  $\Gamma_1, \Gamma_2, \Gamma_3$  and  $E_1, E_2, E_3$  such that  $\Gamma_\mu^2 = E_\mu; \Gamma_3^2 = 1; \{\Gamma_\mu, \Gamma_\nu\} = 0, \mu \neq \nu; \{\Gamma_\mu, \Gamma_3\} = 0; [E_\mu, \Gamma_\nu] = [E_\mu, \Gamma_3] = [E_\mu, E_\nu] = 0, \forall \mu, \nu; \Delta(E_\mu) = E_\mu \otimes \mathbf{1} + \mathbf{1} \otimes E_\mu; S(E_\mu) = -E_\mu, \epsilon(E_\mu) = 0, \Delta(\Gamma_\mu) = \Gamma_\mu \otimes \mathbf{1} + \Gamma_3 \otimes \Gamma_\mu; S(\Gamma_\mu) = \Gamma_\mu \Gamma_3; \epsilon(\Gamma_\mu) = 0; \Delta(\Gamma_3) = \Gamma_3 \otimes \Gamma_3; S(\Gamma_3) = \Gamma_3; \epsilon(\Gamma_3) = 1; \text{ where } \mu, \nu = 1, 2.$

The quantum deformation of  $CH(2)$ ; i.e.  $CH_q(2)$ , is carried out by only



transforming within this, the following terms

$$\Gamma_\mu^2 = [E_\mu]_q = \frac{q^{E_\mu} - q^{-E_\mu}}{q - q^{-1}}; \quad \Delta\Gamma_\mu = \Gamma_\mu \otimes q^{-E_\mu/2} + q^{E_\mu/2} \Gamma_3 \otimes \Gamma_\mu.$$

$\widehat{CH_q(2)}$ , the affinization of  $CH_q(2)$ , is generated by  $E_\mu^{(i)}, \Gamma_\mu^{(i)}$  ( $i = 0, 1; \mu = 1, 2$ ) and  $\Gamma_3$  satisfying  $CH_q(2)$  for each value of  $i$ . Here  $q$  is any complex number.

A two dimensional irrep of  $\widehat{CH_q(2)}$  is labelled by  $(z, \lambda_x, \lambda_y) \in \mathbf{C}^3$  and reads [10];

$$\Gamma_x^{(0)} = \left( \frac{\lambda_x^{-1} - \lambda_x}{q - q^{-1}} \right)^{1/2} \begin{pmatrix} 0 & z^{-1} \\ z & 0 \end{pmatrix}, \quad \Gamma_y^{(0)} = \left( \frac{\lambda_y^{-1} - \lambda_y}{q - q^{-1}} \right)^{1/2} \begin{pmatrix} 0 & -iz^{-1} \\ iz & 0 \end{pmatrix},$$

$$\Gamma_x^{(1)} = \left( \frac{\lambda_x - \lambda_x^{-1}}{q - q^{-1}} \right)^{1/2} \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}, \quad \Gamma_y^{(1)} = \left( \frac{\lambda_y - \lambda_y^{-1}}{q - q^{-1}} \right)^{1/2} \begin{pmatrix} 0 & -iz \\ iz^{-1} & 0 \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q^{E_x^{(0)}} = \lambda_x^{-1}, \quad q^{E_y^{(0)}} = \lambda_y^{-1}, \quad q^{E_x^{(1)}} = \lambda_x \quad \text{and} \quad q^{E_y^{(1)}} = \lambda_y.$$

Let us define the map  $\alpha : \sum_\rho \sum_\mu (\Gamma^{(i)\mu})_\nu^\rho \otimes \sigma_\mu \rightarrow \sigma_\nu$  where  $\mu, \nu, \rho = 1, 2$  and

$\alpha : 1 \otimes \sigma_3 \rightarrow \sigma_3$ , being  $\{\sigma_k\}$  a basis for the Lie algebra of  $su(2)$ ; i.e.

the Pauli matrices. Before this map is applied,  $\sigma_i^2 = 1$  and  $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ ;  $i, j = 1, 2, 3$ . After applying the map  $(\alpha_{\sigma_i})^2 = 0$ ,  $(\alpha_{\sigma_3})^2 = 1$ ,  $\{\alpha_{\sigma_1}, \alpha_{\sigma_3}\} = 0$  and  $\{\alpha_{\sigma_i}, \alpha_{\sigma_j}\} = 4(1 \otimes 1)$ ;  $i, j = 1, 2$ .

From this, it is straightforward to verify the  $\widehat{CH}_q(2)$ -invariance of the Clifford product in the extended Cartan map structure [11] which form a quaternion algebra.

On the other hand,  $H$  acts adjointly on  $A$  if  $\alpha(h \otimes \Gamma) = h_{(1)}\Gamma Sh_{(2)}$  for any  $\Gamma \in A$ . Actually in general  $f(\alpha(h \otimes \Gamma)) = f(h_{(1)})\Gamma f(Sh_{(2)})$  for any  $f$  algebra map. Here  $S : H \rightarrow H$  is the antipode map. For  $A = Cl_{3,1}$  and  $H = SL_q(2)$  the adjoint action looks like  $\alpha(a_{ij} \otimes \Gamma) = a_{ij(1)}\Gamma S(a_{ij(2)})$ , for any  $\Gamma \in Cl_{3,1}$ .

$Cl_Q$  is a  $Z_2$ -graded algebra; i.e.  $Cl_Q = Cl_Q^0 \oplus Cl_Q^1$ , where  $\Lambda^{(r)}(Q)\Lambda^{(s)}(Q) = \Lambda^{(r+s)}(Q)$  and  $1_{Cl_Q} = \Lambda^{(0)}(Q) = k$ . Therefore, any Clifford algebra can be considered as a twisted tensor product of the  $Z_2$ -graded algebras. Since  $Z_2$  is a unital semigroup then  $Cl_Q$  does not induce a  $k(Z_2)$ -module algebra, which is the set of functions on  $Z_2$  with values in  $k$ . This is because the antipode map cannot be defined.

Similarly for  $A$  being a coalgebra we can require the action of  $H$  to respect the coalgebra structure; i.e.

$$\alpha(h \otimes a)_{(1)} \otimes \alpha(h \otimes a)_{(2)} = \alpha(h_{(1)} \otimes a_{(1)}) \otimes \alpha(h_{(2)} \otimes a_{(2)}) \quad \text{and}$$

$$\epsilon \alpha(h \otimes a) = \epsilon(h) \epsilon(a)$$

Since  $\mathcal{Cl}_Q$  is not a coalgebra we do not require this condition to hold.

Dual to the notion of modules is the notion of comodules. Since they are dual their diagrams are obtained by reversing arrows. In other words  $V$  is a left- $H$ -comodule if there is a map  $\beta : V \rightarrow H \otimes V$  such that  $(id \otimes \beta) \circ \beta = (\Delta \otimes id) \circ \beta$  and  $(\epsilon \otimes id) \circ \beta = 1_H \otimes id$ . See Figure 1.

The collection of left  $H$ -comodules is denoted by  ${}^H M$  and the collection of right comodules is denoted by  $M^H$ . The three cases of quantum symmetry above presented can easily be reversed to obtain comodule structures.

Let  $H$  be a finite dimensional Hopf algebra and  $H^*$  its dual. It is well known that there is a one-to-one correspondence between left  $H$ -modules and right  $H^*$ -comodules, left  $H$ -module coalgebras and right  $H^*$ -comodule

coalgebras, left  $H^*$ -module algebras and right  $H^*$ -comodule algebras.

Let  $(\alpha_1, V_1)$  and  $(\alpha_2, V_2)$  be any two representations of the algebra part of  $H$  in the vector spaces  $V_1$  and  $V_2$ .

A *category*  $C$  is a collection of objects  $Ob(C)$ , and a set  $Mor(X, Y)$  for each  $X, Y \in Ob(C)$ . The latter are called morphisms and  $Mor(X_1, Y_1)$ ,  $Mor(X_2, Y_2)$  are disjoint unless  $X_1 = X_2$  and  $Y_1 = Y_2$ . They should have properties analogous to those of maps from  $X$  to  $Y$  that respect the structure on  $X$  and  $Y$ . The representations of an algebra form a category; i.e., the category of the Clifford representations of  $C_q(3, 1)$ . Indeed, an object in this category is a pair  $(\alpha, V)$ . The morphisms  $Mor((\alpha_1, V_1), (\alpha_2, V_2))$  in this category are the interwiners, namely  $\phi \in Lin_k(V_1, V_2)$ , these are the  $k$ -linear maps from  $V_1, V_2$  to  $k$  such that

$$\alpha_2(\phi(v)) = \phi(\alpha_1(v)) , \quad \forall v \in V_1$$

The so called reflection equations for quantum groups can be written in terms of interwiners [12].

A map between categories  $F : C_1 \rightarrow C_2$  is called a *covariant functor* if to each object  $X \in Ob(C_1)$  it assigns an object  $F(X) \in Ob(C_2)$  and to each morphism  $\phi \in Mor(X, Y)$  it assigns a morphism  $F(\phi) \in Mor(F(X), F(Y))$  such that  $F(\phi_1 \circ \phi_2) = F(\phi_1) \circ F(\phi_2)$ . For a *contravariant functor*  $F : C_1 \rightarrow C_2$ , also assigns an object  $F(X) \in Ob(C_2)$  to each object  $X \in Ob(C_1)$  but assigns an element  $F(\phi) \in Mor(F(Y), F(X))$  for each  $\phi \in Mor(X, Y)$  such that  $F(\phi_1 \circ \phi_2) = F(\phi_2) \circ F(\phi_1)$ .

On the other hand a *natural transformation*  $\Phi : F_1 \rightarrow F_2$  between two functors  $F_1, F_2 : C_1 \rightarrow C_2$  is a map that assigns to each object  $X \in Ob(C)$  a morphism  $\Phi_X \in Mor(F_1(X), F_2(X))$  such that for any morphism  $\phi \in Mor(X, Y)$  in  $C_1$ ,  $\Phi_Y \circ F_1(\phi) = F_2(\phi) \circ \Phi_X$  and similarly if  $F_1$  and  $F_2$  are contravariant.

A natural transformation  $\Phi$  is called a *natural equivalence* of functors if each map  $\Phi_X$  is an isomorphism. The maps  $\Phi_X$  in this case are also said to be functorial isomorphisms.

A category  $(C, \tilde{\otimes}, \underline{1})$ , is called monoidal if it has a product functor  $\tilde{\otimes} :$

$C \times C \rightarrow C$  and a unit object  $\underline{1} \in Ob(C)$  such that;

a) The two functors  $C \times C \times C \rightarrow C$  given by  $\tilde{\otimes}(\tilde{\otimes})$  and  $(\tilde{\otimes})\tilde{\otimes}$  are naturally equivalent; i.e. in addition to  $\tilde{\otimes}$ , there are functorial isomorphisms

$$\Phi_{X,Y,Z} : X\tilde{\otimes}(Y\tilde{\otimes}Z) \rightarrow (X\tilde{\otimes}Y)\tilde{\otimes}Z.$$

b)  $X\tilde{\otimes}Y\tilde{\otimes}Z\tilde{\otimes}W$  is associative.

c) Finally, the functors  $C \rightarrow C$  given by  $X \rightarrow X\tilde{\otimes}\underline{1}$  and  $X \rightarrow \underline{1}\tilde{\otimes}X$  should be naturally equivalent to the identity functor; i.e. there are functorial isomorphisms  $X\tilde{\otimes}\underline{1} \rightarrow X$  and  $\underline{1}\tilde{\otimes}X \rightarrow X$ .

Consider  $({}_H M, \tilde{\otimes}, \phi, \underline{1})$ , where  ${}_H M$  is the category of algebra representations of Hopf algebras,  $\tilde{\otimes}$  is defined on  ${}_H M$  by  $V_1\tilde{\otimes}V_2 = V_1 \otimes V_2$  as vector spaces,  $\alpha(h \otimes (v_1\tilde{\otimes}v_2)) = \alpha(h_{(1)} \otimes v_1)\tilde{\otimes}\alpha(h_{(2)} \otimes v_2) \forall v_1\tilde{\otimes}v_2 \in V_1\tilde{\otimes}V_2$  and where the unit is defined by the trivial representation on  $k$ . Then  ${}_H M$  is a monoidal category.

In this case, there is a functor  $F : {}_H M \rightarrow Vec$  that assigns to  $(\rho, V)$  the vector space  $V$  and to a morphism  $\phi \in Mor((\rho_1 V_1), (\rho_2, V_2))$  the interwiner

viewed just as a linear map  $\phi \in Mor_{Vec}(V_1, V_2) = Lin_k(V_1, V_2)$ . This is called the forgetful functor and respects the monoidal structure.

We construct the Clifford algebra  $Cl_Q$  as the quotient algebra of  $\otimes V$  with respect to the two-sided ideal  $I(Q)$  generated by the elements  $X \otimes X - Q(X)$  where  $X \in V$  and  $Q$  is a quadratic form on  $V$  and we consider monoidal tensor categories.

A tensor category  $(C, \tilde{\otimes}, \phi, \Psi, \underline{1})$  is a monoidal category  $(C, \tilde{\otimes}, \phi, \underline{1})$  such that the two functors  $C \times C \rightarrow C$  given by  $X \tilde{\otimes} Y$  and  $Y \tilde{\otimes} X$  are naturally equivalent; namely there exist functorial isomorphisms  $\Psi_{X,Y} : X \tilde{\otimes} Y \rightarrow Y \tilde{\otimes} X$ . From this, the Clifford category is induced.

A quasitriangular Hopf algebra is a pair  $(H, R)$  where  $H$  is a Hopf algebra and  $R \in H \otimes H$  is invertible and obeys  $(\Delta \otimes id)R = R_{13}R_{23}$ ,  $(id \otimes \Delta)R = R_{13}R_{12}$ ,  $\tau \circ \Delta h = R(\Delta h)R^{-1}$ ,  $\forall h \in H$ . Here  $\tau : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$  is the twist map.  $(H, R)$  is called triangular if, in addition,  $\tau(R^{-1}) = R$ . The notation used is  $R = \sum R^{(1)} \otimes R^{(2)}$ . For  $(H, R)$  as a triangular Hopf algebra  $\Psi_{V_1, V_2} : V_1 \tilde{\otimes} V_2 \rightarrow V_2 \tilde{\otimes} V_1$  and  $\Psi(v_1 \tilde{\otimes} v_2) = \tau \circ (\rho_1 \otimes \rho_2)(R)(v_1 \tilde{\otimes} v_2)$  makes  ${}_H M$  into a

tensor category which, modulo  $I(Q)$ , corresponds to the Clifford category  $Cl_Q$ .

A tensor category  $(C, \tilde{\otimes}, \phi, \Psi, \underline{1})$  has an object called “internal hom” if the contravariant functors  $F_{X,Y} = Mor(\tilde{\otimes} X, Y)$  (i.e. that send  $Z$  to the set  $Mor(Z \tilde{\otimes} X, Y)$ ) are each representable. In this case the representing object in  $C$ , the internal hom, is denoted  $\underline{Hom}(X, Y)$ .

Let  $H$  be a triangular Hopf algebra with bijective antipode, then its finite dimensional algebra representations  $(M^{f.d.}, \tilde{\otimes}, \underline{1})$  is a rigid tensor category (therefore this induces a Clifford tensor category) with internal hom defined by  $\underline{Hom}(V_1, V_2) = Lin_k(V_1, V_2)$  and the map  $\hat{\psi} \in Mor(Z, \underline{Hom}(X, Y))$  for which  $(\hat{\psi} \tilde{\otimes} id) \circ ev_{X,Y} = \psi$  where  $ev_{X,Y} : \underline{Hom}(X, Y) \tilde{\otimes} X \rightarrow Y$  is the evaluation map.  $\underline{Hom}(X, Y)$  is like “linear maps from  $X$  to  $Y$ ” and  $ev_{X,Y}$  “applies” this to an element of  $X$  to obtain an element of  $Y$ .

Suppose that  $C$  is a small abelian ( $k$ -linear) rigid tensor category, and  $F$  is a ( $k$ -linear, exact, faithful) monoidal functor. Then, essentially, there is a triangular  $H$  such that  $C$  is equivalent to  ${}_H M$ . If in addition  $F$  coin-



cides with the commutativity constrain in  $Vec$ . Then the reconstructed  $H$  is cocommutative i.e. essentially of the form  $kG$ . Here  $G$  is a group and  $kG$  denote the vector space with basis  $G$ .

We can make the axioms of a rigid quasitensor category even more explicit by choosing basis for each of the spaces  $V_j$ . Thus, let  $\{e_m^j\}$  be a basis for  $V_j$ , then the general decomposition  $V_{j_1} \tilde{\otimes} V_{j_2} \cong \tilde{\oplus}_j \gamma_{j_1 j_2}^j \tilde{\otimes} V_j$  takes the form

$$e_{m_1}^{j_1} \tilde{\otimes} e_{m_2}^{j_2} = \sum_{j,m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \tilde{\otimes} e_m^{j(j_1 j_2)}$$

where  $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \in \gamma_{j_1 j_2}^j$  are called the generalized Clebsch-Gordan coefficients(CGC).

Recall that in the ordinary Lorentz group, the spinor and vector representations are connected by the corresponding Clebsch-Gordan coefficients, or  $\gamma$  matrix in a more familiar terminology. The superfix  $^{(j_1 j_2)}$  on the vectors  $e_m^j$  is to remind us that they are being viewed in  $V_{j_1} \tilde{\otimes} V_{j_2}$  according to the isomorphism above described. Thus, CGC transform the basis  $\{e_m^{j(j_1 j_2)}\}$  to the standard basis  $\{e^{j_1} \tilde{\otimes} e^{j_2}\}$  of  $V_{j_1} \tilde{\otimes} V_{j_2}$ .

We can take advantage of this and define  $q$ -deformed Fierz identities also.

Let  $\psi_a$  be a Majorana  $q$ -spinor [6]; namely  $\psi_a = \begin{pmatrix} Z_a^k \\ (\bar{Z}_a \varepsilon^{-1})^{\bar{k}} \end{pmatrix}$ , where  $a = 1, 2$  and  $Z_a$  is a  $q$ -spinor and  $\varepsilon$  a metric [6]. Then we define the following currents;

$$J_q = \frac{1}{q\sqrt{Q}} \bar{\psi}_1 \psi_2 \quad , \quad J_q^\mu = \frac{1}{q\sqrt{Q}} \bar{\psi}_1 \gamma_q^\mu \psi_2 \quad , \quad J_q^5 = \frac{1}{q\sqrt{Q}} \bar{\psi}_1 \gamma_q^5 \psi_2,$$

$$J_q^{\mu\nu} = \frac{1}{q\sqrt{Q}} \bar{\psi}_1 \gamma_q^\mu \gamma_q^\nu \psi_2 \quad \text{and} \quad J_q^{\mu\nu\tau} = \frac{1}{q\sqrt{Q}} \bar{\psi}_1 \gamma_q^\mu \gamma_q^\nu \gamma_q^\tau \psi_2,$$

where  $\mu, \nu, \tau = 0, +, -, 3$  correspond to the indices in  $\gamma_q^\mu$  previously defined,

and  $\gamma_q^5 = \gamma_q^0 \gamma_q^+ \gamma_q^- \gamma_q^3$ .

There exists a finite number of well defined relationships among these  $q$ -Fierz identities. We can easily find the following relations among them;

$$J_q^{53} = -q^2 J_q^{50} \quad , \quad J_q^{0-} = -J_q^{+3} \quad , \quad J_q^{35} = J_q^{05} \quad , \quad J_q^{-0} = q^{-2} J_q^{3+},$$

$$J_q^{0+} = q^2 J_q^{-3} \quad , \quad J_q^{5+} = J_q^{+-} \quad \text{and} \quad J_q^{+0} = -J_q^{3-}.$$

Even more, some other relations appear among these  $q$ -deformed currents.

For example  $q^4 J_q^2 - (J_q^{03})^2 = Q(1 - q^{-4})(J_q^5)^2$  where the following particular commutation relation between the  $q$ -spinor and its adjoint has been imposed;

$$Z(\epsilon \bar{Z}) = k \hat{R}_q(\epsilon \bar{Z}) Z, \text{ being } k \text{ a real number and } \hat{R}_q = q \sum_\rho e_\rho^\rho \otimes e_\rho^\rho +$$

$\sum_{\rho \neq \sigma} e_\rho^\sigma \otimes e_\sigma^\rho + (q - q^{-1}) \sum_{\rho < \sigma} e_\rho^\rho \otimes e_\sigma^\sigma$ , where  $\rho, \sigma = 1, 2$  and  $e_\rho^\sigma$  is the basis of the matrix. This comes as a natural consequence of the particular braided algebra (reflection equation) chosen [12].

### 3 Summary and Conclusions

Quantum algebras are relatively new mathematical structures which provide an exciting generalization of the concept of symmetry. Quantum spin chains provide the simplest examples of physical systems which have a quantum algebra as invariance.

Assume that  $\mathcal{Cl}_{3,1}$  is involved in a theory describing a physical reality. We know that any physical theory describes well only a limited class of phenomena, for the phenomena beyond this class one must modify the theory. In certain cases such a modification consists in introducing one fundamental constant  $q$  (small parameter) in the new more general theory (f.e. a quantum theory). Within this new theory,  $\mathcal{Cl}_{3,1}$  retains its validity only in the approximate sense. The old theory can be recovered in a limit value for  $q$ .

Studying all possible deformations of spacetime Clifford algebras one may discover ways leading to more general theories that might better describe the reality.

In this context we are presenting a short review on the theory representation of Hopf algebras on Clifford algebras. Some examples are also given.

We intend to search  $C_q(3, 1)$  as a symmetry acting on Clifford spacetime paving the way to a new approach to quantized spacetime. It turns out that upon acting on Clifford spacetime, quantum (i.e.  $q$ -deformed) Clifford spacetime algebra induces a change of metric. From this, we assume any quantum symmetry acting on spacetime as performed by a map whose action can be expressed in terms of well defined fluctuations of metric.

In the second example we propose  $\widehat{CH}_q(2)$  acting as a symmetry of  $\mathfrak{su}(2)$ ; i.e. as a quantum symmetry of the isospin space. Again the result can be expressed in terms of particular fluctuations in the metric of the Lie algebra.

Actually these two cases remind us of the map which can be exhibited between the generators obeying  $SU_q(2)$  commutation relations and the ones

satisfying  $\mathfrak{su}(2)$  algebra. The properly modified open anisotropic Hiesenberg chain possesses  $SU_q(2)$  invariance as a higher non-manifest symmetry. We are thinking of  $C_q(3, 1)$  as a higher non-manifest symmetry of the spacetime that realizes quantum symmetries of Clifford algebras.

As done in the multivectorial generalization of the Cartan map, we propose comodule maps to explicitly show isotopic spaces where this symmetry is realized. Here we propose to study the category of Clifford representations of  $q$ -deformed (in some cases also Hopf algebras) structures.

In this paper we show how the Fierz identities, written in terms of products of generalized multivectorial Cartan maps, have  $C_q(3, 1)$  as a higher non-manifest symmetry as well as  $\mathfrak{su}(2)$  has  $\widehat{CH}_q(2)$  as a non-manifest symmetry also.

Since the  $c < 1$  unitary rational conformal theories can be projected out from the  $SU_q(2)$ -invariant spin  $1/2$  chain for  $q$  a primitive root of unity (in the thermodynamic limit) we expect  $C_q(3, 1)$ -invariant Fierz identities to be an extension of sensitive meaningful magnitudes in field theory. Even

more, the representation theory of  $SU_q(2)$  has interesting connections to the solutions of closed spin  $1/2$  anisotropic Heisenberg chain. We think that the representation theory of  $C_q(3, 1)$  may have connections to some interesting lattice model for spacetime Clifford algebra. Some  $q$ -deformed Fierz identities are also studied.

The spin 1 Faddeev-Zamolodchikov chain has a class of boundary terms, which makes the model  $SU_q(2)$ -symmetric. We wonder what kind of discrete model with a class of boundary terms corresponds to  $C_q(3, 1)$ -symmetric Fierz identities. It would be interesting to find physical systems with this symmetry.

It has been shown [13] that a Hamiltonian is actually already invariant under the classical group by virtue of its invariance under the quantum group. Since Fierz identities are  $C_q(3, 1)$  invariant they certainly are  $\mathcal{C}\ell_{3,1}$  invariant. This last symmetry can be realized in a variety of ways, in particular the results reported by Rodríguez-Romo [7], [8], [9].

Generalized statistics of physical particles are closely connected with the

invariance under quantum groups. This invariance provides the possibility to construct parafermions possessing generalized statistics which interpolates the physical particles [5]. This means in our case that the Fierz identities already constructed for fermions can be generalized to parafermions in a straightforward manner.

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## 5 Figure Captions

Figure 1. Axioms for left  $H$ -module and  $H$ -comodule structures written as diagrams.

$$\begin{array}{ccc}
H \otimes V & \xleftarrow{id \otimes \alpha} & H \otimes H \otimes V \\
\downarrow \alpha & & \cdot \otimes id \downarrow \\
V & \xleftarrow{\alpha} & H \otimes V
\end{array}
\qquad
\begin{array}{ccc}
H \otimes V & \xrightarrow{id \otimes \beta} & H \otimes H \otimes V \\
\uparrow \beta & & \Delta \otimes id \uparrow \\
V & \xrightarrow{\beta} & H \otimes V
\end{array}$$
  

$$\begin{array}{ccc}
H \otimes V & \xleftarrow{\eta \otimes id} & V \\
\downarrow \alpha & \swarrow & 1_H \otimes id \\
V & & 
\end{array}
\qquad
\begin{array}{ccc}
H \otimes V & \xrightarrow{\epsilon \otimes id} & V \\
\uparrow \beta & \nearrow 1_H \otimes id & \\
V & & 
\end{array}$$

Figure 1.



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